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The frictionless rolling contact of a rigid circular cylinder on a semi-infinite granular material

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Abstract. The resulting flow and deformation of a semi-infinite granular material under a rolling, smooth rigid circular cylinder is investigated using a perturbation method. Based on the double-shearing theory of granular flow, complete stress and velocity fields, resistance to rolling and the permanent displacement of surface particles are determined to first order; when the internal friction angle is zero, the solutions reduce to those obtained in the corresponding analysis for Tresca or von-Mises materials. The solution scheme and the double-shearing model for granular flow both find their origins in the work of A.J.M. Spencer.

Key words: rolling contact, granular materials, double-shearing theory.

1. Introduction

Perturbation methods have been used to obtain approximate solutions for a variety of problems in metal plasticity. One notable example is that originally introduced by A.J.M. Spencer to solve a series of problems involving Tresca and von-Mises type materials [1]–[5] by perturbation on the zero-order characteristics. This procedure was later adopted by Marshall [6] to solve the steady state, frictionless rolling contact of a rigid circular cylinder on a semi-infinite rigid-plastic solid. In this paper, we use Spencer's perturbation method to solve the analogous contact problem for an incompressible granular material obeying the Mohr–Coulomb yield criterion and the kinematic double-shearing theory of Spencer [7]. The resulting stresses in the material due to gravity are considered small compared with those generated by the rolling cylinder, and furthermore it is assumed that the rolling velocity is sufficiently slow. Consequently, both inertial and body forces may be neglected. Under these conditions, the resistance to rolling and the permanent displacement of particles on the surface of the granular material are determined. Finally, the aim of this work is to serve as a prelude to the analysis of the frictional rolling contact problem.

Rolling contact problems involving granular materials have remained largely unsolved despite their prominence in a host of industrial and engineering processes (*e.g.*, in rigid wheel-soil interaction systems which are of interest in the area of off-road vehicle engineering [8]). This is largely due to the complexities of granular behavior which raise more difficulties than those of solids. In particular, there is a relatively large collection of work on establishing the stresses and deformations of semi-infinite solid bodies under a rolling rigid cylinder [9, Chapters 8–9]; these include the important contributions by Collins [10–11] on frictional contact for rigid-plastic Tresca or von-Mises type materials (which are appropriate for metals, or, the highly idealized case of a purely cohesive and incompressible soil), and by Hunter [12] for viscoelastic materials (which are more representative of rubber or polymers). This study provides the foundation upon which an equivalent body of knowledge for rolling contact with granular materials may be established. In this context, the solutions by Marshall [6] for the corresponding contact problem for a rigid-plastic solid are recovered here as special cases.

2. Plane strain double-shearing theory for incompressible granular materials

The notations adopted in this section follow those in Hill and Wu [13]. When inertial and body forces are neglected, the equilibrium equations are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \qquad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0. \tag{1}$$

The yield condition is given by the Mohr–Coulomb criterion

$$\frac{\sigma_1 - \sigma_3}{2} = c\ \cos(\delta) - \frac{\sigma_1 + \sigma_3}{2}\ \sin(\delta), \tag{2}$$

with c and δ being the cohesion and angle of internal friction of the material, and the principal stresses σ_i $(i = 1, 2, 3)$ are in the order of $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$. If the new stress variables

$$p = -\frac{\sigma_1 + \sigma_3}{2}, \qquad q = \frac{\sigma_1 - \sigma_3}{2}, \tag{3}$$

are introduced the Mohr–Coulomb yield criterion becomes

$$q = c\ \cos(\delta) + p\ \sin(\delta). \tag{4}$$

If ψ is the angle between the major principal stress σ_1 and x-axis, then the standard relations between the stress components σ_{ij}, the principal stresses σ_i, and the yield condition (4) lead to the following stress components in the plastic region

$$\begin{aligned}
\sigma_{xx} &= \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2}\ \cos(2\psi) \\
&= -p + q\cos(2\psi) = c\ \cot(\delta) - q\,[\mathrm{cosec}(\delta) - \cos(2\psi)], \\
\sigma_{yy} &= \frac{\sigma_1 + \sigma_3}{2} - \frac{\sigma_1 - \sigma_3}{2}\ \cos(2\psi) \\
&= -p - q\cos(2\psi) = c\ \cot(\delta) - q\,[\mathrm{cosec}(\delta) + \cos(2\psi)], \\
\sigma_{xy} &= \frac{\sigma_1 - \sigma_3}{2}\sin(2\psi) = q\ \sin(2\psi).
\end{aligned} \tag{5}$$

Substituting relations (5) in (1) and making use of the transformations

$$\begin{aligned}
\cos(\delta)\frac{\partial}{\partial x} &= \sin\left(\psi + \frac{\pi}{4} + \frac{\delta}{2}\right)\frac{\partial}{\partial s_\alpha} - \sin\left(\psi - \frac{\pi}{4} - \frac{\delta}{2}\right)\frac{\partial}{\partial s_\beta}, \\
\cos(\delta)\frac{\partial}{\partial y} &= -\cos\left(\psi + \frac{\pi}{4} + \frac{\delta}{2}\right)\frac{\partial}{\partial s_\alpha} + \cos\left(\psi - \frac{\pi}{4} - \frac{\delta}{2}\right)\frac{\partial}{\partial s_\beta},
\end{aligned} \tag{6}$$

we obtain the governing equations for q and ψ

$$\cot(\delta)\frac{\partial q}{\partial s_\alpha} + 2q\frac{\partial \psi}{\partial s_\alpha} = 0, \qquad \cot(\delta)\frac{\partial q}{\partial s_\beta} - 2q\frac{\partial \psi}{\partial s_\beta} = 0. \tag{7}$$

Here $\partial/\partial s_\alpha$ and $\partial/\partial s_\beta$ are directional derivatives along the so-called α- and β-stress characteristic lines which are, respectively,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\left(\psi - \frac{\pi}{4} - \frac{\delta}{2}\right) \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \tan\left(\psi + \frac{\pi}{4} + \frac{\delta}{2}\right). \tag{8}$$

Relative to these characteristic lines, the double-shearing theory yields the following governing equations for the velocity components v_α and v_β in the directions of the α- and β-lines,

$$\begin{aligned} \cos(\delta)\frac{\partial v_\alpha}{\partial s_\alpha} &= v_\beta\left[\frac{\partial \psi}{\partial s_\alpha} + \sin(\delta)\frac{\partial \psi}{\partial s_\beta}\right] + \psi_t \sin(\delta), \\ \cos(\delta)\frac{\partial v_\beta}{\partial s_\beta} &= -v_\alpha\left[\frac{\partial \psi}{\partial s_\beta} + \sin(\delta)\frac{\partial \psi}{\partial s_\alpha}\right] - \psi_t \sin(\delta), \end{aligned} \tag{9}$$

where $\psi_t = \partial\psi/\partial t$ is the partial derivative with respect to time t of the angle ψ.

Since

$$\tan(2\psi) = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}},$$

the term ψ_t in (9) represents the effects of the stress-rates $\partial\sigma_{ij}/\partial t$ on the velocity. Thus, before the velocity field can be solved, the stress-rate field must first be determined. The stress-rate field, in terms of $q_t = \partial q/\partial t$ and ψ_t, must satisfy the equilibrium equations

$$\begin{aligned} \cot(\delta)\frac{\partial q_t}{\partial s_\alpha} + 2q_t\frac{\partial \psi}{\partial s_\alpha} + 2q\frac{\partial \psi_t}{\partial s_\alpha} + 2\psi_t \mathrm{cosec}(\delta)\frac{\partial q}{\partial s_\beta} &= 0, \\ \cot(\delta)\frac{\partial q_t}{\partial s_\beta} - 2q_t\frac{\partial \psi}{\partial s_\beta} - 2q\frac{\partial \psi_t}{\partial s_\beta} - 2\psi_t \mathrm{cosec}(\delta)\frac{\partial q}{\partial s_\alpha} &= 0. \end{aligned} \tag{10}$$

These equations are obtained from the partial differentiation of Equations (1) and (5) with respect to time t, and by the transformations given in (6).

In general, for a given set of boundary conditions, q and ψ are first determined from Equations (7), followed by q_t and ψ_t from Equations (10) and, finally, the velocities v_α and v_β from Equations (9).

Assume that the tangent of the surface boundary of the material is inclined at an angle λ with the x-axis. If on part of the boundary, the normal and tangential components of the traction are given by σ_n and τ, then the boundary conditions for q and ψ on this part of boundary are

$$\sigma_n = c\ \cot(\delta) - q[\operatorname{cosec}(\delta) + \cos(2\psi - 2\lambda)],$$
$$\tau = q\sin(2\psi - 2\lambda). \tag{11}$$

On this part of the boundary, the traction rates $\partial\sigma_n/\partial t$ and $\partial\tau/\partial t$ should also be specified and conditions for q_t and ψ_t are

$$\frac{\partial\sigma_n}{\partial t} = -q_t[\operatorname{cosec}(\delta) + \cos(2\psi - 2\lambda)] + 2q(\psi_t - \lambda_t)\ \sin(2\psi - 2\lambda),$$
$$\frac{\partial\tau}{\partial t} = q_t\ \sin(2\psi - 2\lambda) + 2q(\psi_t - \lambda_t)\ \cos(2\psi - 2\lambda). \tag{12}$$

Here $\lambda_t = \partial\lambda/\partial t$ is the rate of rotation of the boundary tangent at the given point. On that part of the boundary where velocity components are given by $v_x = V_x$ and $v_y = V_y$, the conditions for v_α and v_β are then

$$V_x = \cos\left(\psi - \frac{\pi}{4} - \frac{\delta}{2}\right) v_\alpha + \cos\left(\psi + \frac{\pi}{4} + \frac{\delta}{2}\right) v_\beta,$$
$$V_y = \sin\left(\psi - \frac{\pi}{4} - \frac{\delta}{2}\right) v_\alpha + \sin\left(\psi + \frac{\pi}{4} + \frac{\delta}{2}\right) v_\beta. \tag{13}$$

Alternatively, if the normal component of velocity V_n is specified, then the corresponding condition is

$$V_n = v_\alpha\ \sin\left(\psi - \frac{\pi}{4} - \frac{\delta}{2} - \lambda\right) + v_\beta\ \sin\left(\psi + \frac{\pi}{4} + \frac{\delta}{2} - \lambda\right). \tag{14}$$

Across a bounding and/or interfacial α-line, the stresses, the stress rate combination $\cot(\delta)q_t - 2q\psi_t$, and the velocity component v_β must be continuous; on the other hand, the stresses, the stress rate combination $\cot(\delta)q_t + 2q\psi_t$ and the velocity component v_α must be continuous across a bounding and/or interfacial β-line. (The term 'bounding' refers to a line separating the rigid region from the plastic region, while 'interfacial' refers to a line separating two plastic zones.)

3. Spencer's perturbation method

Suppose we know the solution for a body $\mathbf{B}_0$ whose boundary at time t, in parametrical form, is

$$x = x^0(\mu, t), \quad y = y^0(\mu, t)$$

and that the conditions on the boundary are given by σ_n^0, τ^0, $\partial\sigma_n^0/\partial t$, $\partial\tau^0/\partial t$, V_x^0 and V_y^0. Let the known solutions for body $\mathbf{B}_0$ be denoted by q^0, ψ^0, q_t^0, ψ_t^0, v_α^0 and v_β^0. We now attempt to determine the solution for another body $\mathbf{B}$ whose boundary is slightly different from $\mathbf{B}_0$ such that

$$x = x^0(\mu, t) + \varepsilon x'(\mu, t), \qquad y = y^0(\mu, t) + \varepsilon y'(\mu, t),$$

where ε is a small parameter. The conditions on the boundary of **B** are also slightly different from those of $\mathbf{B}_0$, and are given by $\sigma_n^0 + \varepsilon\sigma_n'$, $\tau^0 + \varepsilon\tau'$, $\partial\sigma_n^0/\partial t + \varepsilon\partial\sigma_n'/\partial t$, $\partial\tau^0/\partial t + \varepsilon\partial\tau'/\partial t$, $V_x^0 + \varepsilon V_x'$ and $V_y^0 + \varepsilon V_y'$. Accordingly, we seek solutions for **B** of the form

$$\begin{aligned} q &= q^0 + \varepsilon q' + \varepsilon^2 q'' + \cdots & \psi &= \psi^0 + \varepsilon\psi' + \varepsilon^2\psi'' + \cdots \\ q_t &= q_t^0 + \varepsilon q_t' + \varepsilon^2 q_t'' + \cdots & \psi_t &= \psi_t^0 + \varepsilon\psi_t' + \varepsilon^2\psi_t'' + \cdots \\ v_\alpha &= v_\alpha^0 + \varepsilon v_\alpha' + \varepsilon^2 v_\alpha'' + \cdots & v_\beta &= v_\beta^0 + \varepsilon v_\beta' + \varepsilon^2 v_\beta'' + \cdots . \end{aligned} \tag{15}$$

We obtain the governing equations for each order by substituting (15) in the governing Equations (7), (10) and (9), and noting the following relations between the differentiations to order ε, (Spencer [4])

$$\begin{aligned} \frac{\partial}{\partial s_\alpha} &= \sec(\delta)\left[\cos(\delta - \psi + \psi^0)\frac{\partial}{\partial s_\alpha^0} + \sin(\psi - \psi^0)\frac{\partial}{\partial s_\beta^0}\right] \\ &= \frac{\partial}{\partial s_\alpha^0} + \varepsilon\psi'\left[\tan(\delta)\frac{\partial}{\partial s_\alpha^0} + \sec(\delta)\frac{\partial}{\partial s_\beta^0}\right], \\ \frac{\partial}{\partial s_\beta} &= \sec(\delta)\left[-\sin(\psi - \psi^0)\frac{\partial}{\partial s_\alpha^0} + \cos(\delta + \psi - \psi^0)\frac{\partial}{\partial s_\beta^0}\right] \\ &= \frac{\partial}{\partial s_\beta^0} - \varepsilon\psi'\left[\tan(\delta)\frac{\partial}{\partial s_\beta^0} + \sec(\delta)\frac{\partial}{\partial s_\alpha^0}\right], \end{aligned} \tag{16}$$

and by equating the coefficients of successive powers of ε in each equation to zero. The equations and boundary conditions for the zeroth order have the same form as in (7), (9)–(14), with all the variables replaced by the zeroth order terms. Hence the governing equations for the zeroth order stress, stress-rate and velocity fields are

$$\cot(\delta)\frac{\partial q^0}{\partial s_\alpha^0} + 2q^0\frac{\partial\psi^0}{\partial s_\alpha^0} = 0, \qquad \cot(\delta)\frac{\partial q^0}{\partial s_\beta^0} - 2q^0\frac{\partial\psi^0}{\partial s_\beta^0} = 0, \tag{17}$$

$$\begin{aligned} &\cot(\delta)\frac{\partial q_t^0}{\partial s_\alpha^0} + 2q_t^0\frac{\partial\psi^0}{\partial s_\alpha^0} + 2q^0\frac{\partial\psi_t^0}{\partial s_\alpha^0} + 2\psi_t^0\operatorname{cosec}(\delta)\frac{\partial q^0}{\partial s_\beta^0} = 0, \\ &\cot(\delta)\frac{\partial q_t^0}{\partial s_\beta^0} - 2q_t^0\frac{\partial\psi^0}{\partial s_\beta^0} - 2q^0\frac{\partial\psi_t^0}{\partial s_\beta^0} - 2\psi_t^0\operatorname{cosec}(\delta)\frac{\partial q^0}{\partial s_\alpha^0} = 0, \end{aligned} \tag{18}$$

$$\begin{aligned} \cos(\delta)\frac{\partial v_\alpha^0}{\partial s_\alpha^0} &= v_\beta^0\left[\frac{\partial\psi^0}{\partial s_\alpha^0} + \sin(\delta)\frac{\partial\psi^0}{\partial s_\beta^0}\right] + \psi_t^0\sin(\delta), \\ \cos(\delta)\frac{\partial v_\beta^0}{\partial s_\beta^0} &= -v_\alpha^0\left[\frac{\partial\psi^0}{\partial s_\beta^0} + \sin(\delta)\frac{\partial\psi^0}{\partial s_\alpha^0}\right] - \psi_t^0\sin(\delta), \end{aligned} \tag{19}$$

with the boundary conditions

$$\begin{aligned}\sigma_n^0 &= c\ \cot(\delta) - q^0[\operatorname{cosec}(\delta) + \cos(2\psi^0 - 2\lambda^0)],\\ \tau^0 &= q^0\ \sin(2\psi^0 - 2\lambda^0),\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{\partial\sigma_n^0}{\partial t} &= -q_t^0[\operatorname{cosec}(\delta) + \cos(2\psi^0 - 2\lambda^0)] + 2q^0(\psi_t^0 - \lambda_t^0)\ \sin(2\psi^0 - 2\lambda^0),\\ \frac{\partial\tau^0}{\partial t} &= q_t^0\ \sin(2\psi^0 - 2\lambda^0) + 2q^0(\psi_t^0 - \lambda_t^0)\ \cos(2\psi^0 - 2\lambda^0),\end{aligned} \tag{21}$$

or

$$\begin{aligned}V_x^0 &= \cos\left(\psi^0 - \frac{\pi}{4} - \frac{\delta}{2}\right) v_\alpha^0 + \cos\left(\psi^0 + \frac{\pi}{4} + \frac{\delta}{2}\right) v_\beta^0,\\ V_y^0 &= \sin\left(\psi^0 - \frac{\pi}{4} - \frac{\delta}{2}\right) v_\alpha^0 + \sin\left(\psi^0 + \frac{\pi}{4} + \frac{\delta}{2}\right) v_\beta^0.\end{aligned} \tag{22}$$

The alternative condition, with the normal component of velocity being specified, is

$$V_n^0 = v_\alpha^0\ \sin\left(\psi^0 - \frac{\pi}{4} - \frac{\delta}{2} - \lambda^0\right) + v_\beta^0\ \sin\left(\psi^0 + \frac{\pi}{4} + \frac{\delta}{2} - \lambda^0\right).$$

Across a bounding α^0- (interfacial β^0-)line, the quantities q^0, ψ^0, $\cot(\delta)q_t^0 - 2q^0\psi^0$ ($\cot(\delta)q_t^0 + 2q^0\psi^0$) and v_β^0 (v_α^0) must be continuous.

The governing equations for the first order stress field q' and ψ' are

$$\begin{aligned}&\cot(\delta)\frac{\partial q'}{\partial s_\alpha^0} + 2q^0\frac{\partial\psi'}{\partial s_\alpha^0} + 2q'\frac{\partial\psi^0}{\partial s_\alpha^0} + 2\psi'\operatorname{cosec}(\delta)\frac{\partial q^0}{\partial s_\beta^0} = 0,\\ &\cot(\delta)\frac{\partial q'}{\partial s_\beta^0} - 2q^0\frac{\partial\psi'}{\partial s_\beta^0} - 2q'\frac{\partial\psi^0}{\partial s_\beta^0} - 2\psi'\operatorname{cosec}(\delta)\frac{\partial q^0}{\partial s_\alpha^0} = 0.\end{aligned} \tag{23}$$

The equations for the first order stress-rate field q_t' and ψ_t' are

$$\begin{aligned}&\cot(\delta)\frac{\partial q_t'}{\partial s_\alpha^0} + 2q_t'\frac{\partial\psi^0}{\partial s_\alpha^0} + 2q^0\frac{\partial\psi_t'}{\partial s_\alpha^0} + 2\operatorname{cosec}(\delta)\psi_t'\frac{\partial q^0}{\partial s_\beta^0} + 2\operatorname{cosec}(\delta)\psi_t^0\frac{\partial q'}{\partial s_\beta^0} + 2q'\frac{\partial\psi_t^0}{\partial s_\alpha^0}\\ &\quad + 2q_t^0\frac{\partial\psi'}{\partial s_\alpha^0} + 2\sec(\delta)\psi'\left[\cot(\delta)\frac{\partial q_t^0}{\partial s_\beta^0} + 2\psi_t^0\left(\frac{\partial q^0}{\partial s_\beta^0} + \operatorname{cosec}(\delta)\frac{\partial q^0}{\partial s_\alpha^0}\right)\right] = 0,\\ &\cot(\delta)\frac{\partial q_t'}{\partial s_\beta^0} + 2q_t'\frac{\partial\psi^0}{\partial s_\beta^0} - 2q^0\frac{\partial\psi_t'}{\partial s_\beta^0} - 2\operatorname{cosec}(\delta)\psi_t'\frac{\partial q^0}{\partial s_\alpha^0} - 2\operatorname{cosec}(\delta)\psi_t^0\frac{\partial q'}{\partial s_\alpha^0} - 2q'\frac{\partial\psi_t^0}{\partial s_\beta^0}\\ &\quad - 2q_t^0\frac{\partial\psi'}{\partial s_\beta^0} - 2\sec(\delta)\psi'\left[\cot(\delta)\frac{\partial q_t^0}{\partial s_\alpha^0} + 2\psi_t^0\left(\frac{\partial q^0}{\partial s_\alpha^0} + \operatorname{cosec}(\delta)\frac{\partial q^0}{\partial s_\beta^0}\right)\right] = 0;\end{aligned} \tag{24}$$

for the first order velocity field v'_α and v'_β, we have the equations

$$
\begin{aligned}
&\cos(\delta)\frac{\partial v'_\alpha}{\partial s^0_\alpha} - v'_\beta\left[\frac{\partial \psi^0}{\partial s^0_\alpha} + \sin(\delta)\frac{\partial \psi^0}{\partial s^0_\beta}\right] \\
&\quad + \psi' \sec(\delta)\left\{\cos(\delta)\frac{\partial v^0_\alpha}{\partial s^0_\beta} - v^0_\beta\left[\cos(2\delta)\frac{\partial \psi^0}{\partial s^0_\beta} - \sin(\delta)\frac{\partial \psi^0}{\partial s^0_\alpha}\right]\right\} \\
&\quad + \psi'\psi^0_t \tan(\delta)\sin(\delta) - v^0_\beta\left[\frac{\partial \psi'}{\partial s^0_\alpha} + \sin(\delta)\frac{\partial \psi'}{\partial s^0_\beta}\right] - \psi'_t \sin(\delta) = 0, \\
&\cos(\delta)\frac{\partial v'_\beta}{\partial s^0_\beta} + v'_\alpha\left[\frac{\partial \psi^0}{\partial s^0_\beta} + \sin(\delta)\frac{\partial \psi^0}{\partial s^0_\alpha}\right] \\
&\quad - \psi' \sec(\delta)\left\{\cos(\delta)\frac{\partial v^0_\beta}{\partial s^0_\alpha} + v^0_\alpha\left[\cos(2\delta)\frac{\partial \psi^0}{\partial s^0_\alpha} - \sin(\delta)\frac{\partial \psi^0}{\partial s^0_\beta}\right]\right\} \\
&\quad + \psi'\psi^0_t \tan(\delta)\sin(\delta) + v^0_\alpha\left[\frac{\partial \psi'}{\partial s^0_\beta} + \sin(\delta)\frac{\partial \psi'}{\partial s^0_\alpha}\right] + \psi'_t \sin(\delta) = 0.
\end{aligned}
\tag{25}
$$

Let the equation for the boundary of $\mathbf{B}_0$ be expressed as

$$y = g^0(x, t), \tag{26}$$

then, noting that the angle of the surface tangent to the x-axis may be written as

$$\lambda = \lambda^0 + \varepsilon\lambda' + \varepsilon^2\lambda'' + \cdots,$$

we have

$$\tan(\lambda^0) = \frac{\partial g^0}{\partial x}. \tag{27}$$

Hence, for the boundary of $\mathbf{B}$ expressed in the form

$$y = g^0(x, t) + \varepsilon g'(x, t), \tag{28}$$

it follows that

$$\tan(\lambda^0 + \varepsilon\lambda' + \cdots) = \frac{\partial g^0}{\partial x} + \varepsilon\frac{\partial g'}{\partial x} = \tan(\lambda^0) + \varepsilon\frac{\partial g'}{\partial x}.$$

To order ε, the above yields

$$\lambda' = \cos^2(\lambda^0)\frac{\partial g'}{\partial x}. \tag{29}$$

The first-order conditions on the part of the boundary where the traction and traction rates are specified, are

$$
\begin{aligned}
\sigma_n' &= -q'[\operatorname{cosec}(\delta) + \cos(2\psi^0 - 2\lambda^0)] + 2q^0(\psi' - \lambda')\ \sin(2\psi^0 - \lambda^0),\\
\tau' &= q'\ \sin(2\psi^0 - 2\lambda^0) + 2q^0(\psi' - \lambda')\ \cos(2\psi^0 - 2\lambda^0),\\
\frac{\partial \sigma_n'}{\partial t} &= -q_t'[\operatorname{cosec}(\delta) + \cos(2\psi^0 - 2\lambda^0)] + 2q^0(\psi_t' - \lambda_t')\ \sin(2\psi^0 - \lambda^0)\\
&\quad + 2[q_t^0(\psi' - \lambda') + q'(\psi_t^0 - \lambda_t^0)]\ \sin(2\psi^0 - \lambda^0)\\
&\quad + 4q^0(\psi_t^0 - \lambda_t^0)(\psi' - \lambda')\ \cos(2\psi^0 - 2\lambda^0),\\
\frac{\partial \tau'}{\partial t} &= q_t'\ \sin(2\psi^0 - 2\lambda^0) + 2q^0(\psi_t' - \lambda_t')\ \cos(2\psi^0 - \lambda^0)\\
&\quad + 2[q_t^0(\psi' - \lambda') + q'(\psi_t^0 - \lambda_t^0)]\ \cos(2\psi^0 - \lambda^0)\\
&\quad + 4q^0(\psi_t^0 - \lambda_t^0)(\psi' - \lambda')\ \sin(2\psi^0 - 2\lambda^0).
\end{aligned}
\tag{30}
$$

If on part of the boundary, the velocity components in the x- and y-directions are specified, then the first-order boundary conditions become

$$
\begin{aligned}
V_x' &= v_\alpha'\ \cos\left(\psi^0 - \frac{\pi}{4} - \frac{\delta}{2}\right) + v_\beta'\ \cos\left(\psi^0 + \frac{\pi}{4} + \frac{\delta}{2}\right)\\
&\quad -\psi'\left[v_\alpha^0\ \sin\left(\psi^0 - \frac{\pi}{4} - \frac{\delta}{2}\right) + v_\beta^0 \sin\left(\psi^0 + \frac{\pi}{4} + \frac{\delta}{2}\right)\right],\\
V_y' &= v_\alpha'\ \sin\left(\psi^0 - \frac{\pi}{4} - \frac{\delta}{2}\right) + v_\beta'\ \sin\left(\psi^0 + \frac{\pi}{4} + \frac{\delta}{2}\right)\\
&\quad +\psi'\left[v_\alpha^0\ \cos\left(\psi^0 - \frac{\pi}{4} - \frac{\delta}{2}\right) + v_\beta^0 \cos\left(\psi^0 + \frac{\pi}{4} + \frac{\delta}{2}\right)\right].
\end{aligned}
\tag{31}
$$

If the normal component of velocity is given, then we have

$$
\begin{aligned}
V_n' &= v_\alpha'\ \sin\left(\psi^0 - \frac{\pi}{4} - \frac{\delta}{2} - \lambda^0\right) + v_\beta'\ \sin\left(\psi^0 + \frac{\pi}{4} + \frac{\delta}{2} - \lambda^0\right)\\
&\quad +(\psi' - \lambda')\left[v_\alpha^0 \cos\left(\psi^0 - \frac{\pi}{4} - \frac{\delta}{2} - \lambda^0\right) + v_\beta^0 \cos\left(\psi^0 + \frac{\pi}{4} + \frac{\delta}{2} - \lambda^0\right)\right].
\end{aligned}
\tag{32}
$$

Finally, all of the above first-order equations must satisfy the continuity conditions across the perturbed, bounding α- and interfacial β-lines.

4. Rolling contact of a rigid cylinder on a semi-infinite granular material

As depicted in Figure 1, we consider the fully developed steady-state rolling of a smooth rigid circular cylinder on the surface of a semi-infinite granular material. The material is assumed to be incompressible so that the surface behind the cylinder is at the same level as the undeformed surface in front of the cylinder, and furthermore the volume of 'bulldozed' material immediately ahead of the cylinder remains constant. In accordance with Marshall

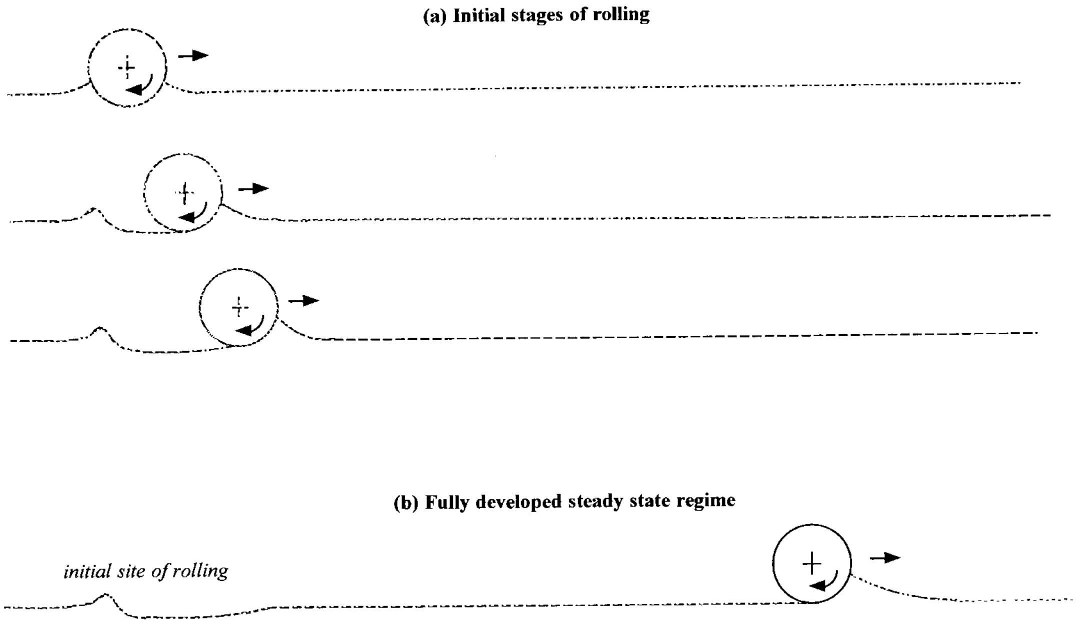

Figure 1. Proposed origin of the bulldozed region found in the fully developed steady state regime: (a) *the initial stages of rolling* – cylinder first indents causing material to rise along both sides of the cylinder, then the cylinder pushes the right-hand material forwards in the direction of travel as it ascends back to the undeformed level; (b) *the fully developed, steady state rolling contact* – cylinder, at the undeformed level, pushing the established bulldozed region. The incompressibility condition requires that the volume of material above the undeformed level (*i.e.* bulldozed material and any raised material from the initial stages) equals the volume displaced below the undeformed level.

[6], the bulldozed material arises at the initial stages of rolling and leaves behind a permanent depression on the surface of the material (see, for example, the experiments with hardened steel balls by Eldridge and Tabor [14], which produced a depression in the roll track situated at the site of initial contact).

Since the contact is frictionless, a translatory force (which must exceed the rolling resistance) and a torque are applied at the centre of the cylinder to allow both forward translation as well as rotation of the cylinder. The resulting deformation which occurs in front of the cylinder centreline is illustrated in Figure 2. Let the time start when the cylinder centreline coincides with the y-axis. At this instant, the portion of the cylinder surface near the origin of the coordinate system Oxy can be approximated by

$$y = \frac{x^2}{(2R)} = \frac{\varepsilon x^2}{(2a)} \equiv \frac{\varepsilon x^2}{b}, \tag{33}$$

where R is the radius of the cylinder and $\varepsilon = a/R$ is small with a being the contact width, *i.e.* the distance from the centreline to the point of separation between the cylinder and the free surface of the material, A’. Thenceforth, the portion of cylinder surface near the surface of the material is given by

$$y = \frac{\varepsilon(x - ut)^2}{b,} \tag{34}$$

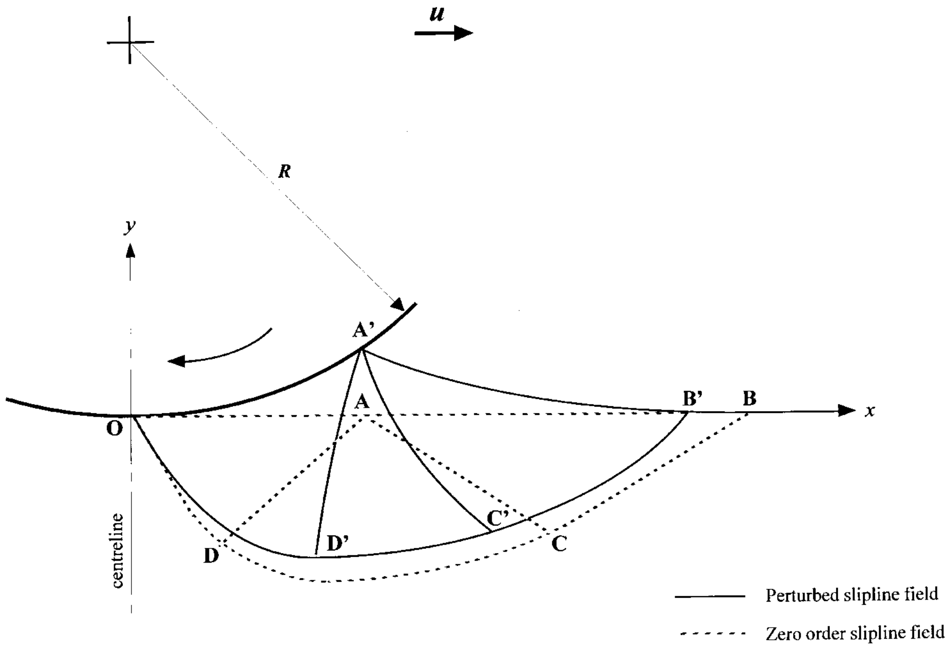

Figure 2. Slipline field for the frictionless rolling contact of a cylinder on an incompressible semi-infinite granular material; in addition to the vertical load W, a torque and a translatory force $(> F)$ acts at the centre of the cylinder which moves forward at speed u. (A change of coordinate axes from Oxy to O'$x'y'$ applies to all discussions from Section 4.1 onwards)

where u is the speed of the cylinder. We attempt to find the stress, stress rates and velocity fields in the plastic region, as well as the profile of the free surface, by the above perturbation method.

4.1. The zeroth-order solutions

The zeroth-order solution is obtained by assuming that the corresponding slipline field is a part of that proposed by Hill [15] for the indentation of a semi-infinite body by a smooth flat punch whose half-width is equal to the contact width a. Accordingly, the slipline field for $\mathbf{B}_0$ consists of two constant stress regions ABC and OAD connected by a spiral fan ADC with centre at A which lie on the undisturbed surface exactly below the point of separation A' (see Figure 2). It follows that

$$g^0(x,t) = 0, \qquad \lambda^0 = \lambda_t^0 = 0$$

on the surface OAB. The cylinder rolls horizontally, so the zeroth-order boundary conditions on the contact surface OA are

$$\psi^0 = 0, \qquad \psi_t^0 = 0, \qquad v_\alpha^0 - v_\beta^0 = 0.$$

On the traction free surface AB, the conditions are

$$\psi^0 = \frac{\pi}{2}, \qquad q^0 = c\,\frac{\cos(\delta)}{1-\sin(\delta)}, \qquad \psi_t^0 = q_t^0 = 0.$$

The zeroth-order solutions satisfying these boundary conditions and the continuity conditions on the bounding α^0-line ODCB and the interfacial β^0-lines AD and AC are (see Spencer [2]):

$$\begin{aligned}
&\psi^0 = \frac{\pi}{2}, \qquad q^0 = c\,\frac{\cos(\delta)}{1-\sin(\delta)} \equiv q_I^0, \quad \text{in ABC},\\
&\psi^0 = \theta + \frac{\pi}{4} - \frac{\delta}{2},\\
&q^0 = c\,\frac{\cos(\delta)}{1-\sin(\delta)}\,\exp\left[2\left(\frac{\pi}{4}+\frac{\delta}{2}-\theta\right)\tan(\delta)\right] \equiv q_{II}^0, \quad \text{in ACD},\\
&\psi^0 = 0, \qquad q^0 = c\,\frac{\cos(\delta)}{1-\sin(\delta)}\,\exp[\pi\ \tan(\delta)] \equiv q_{III}^0, \quad \text{in OAD},\\
&\psi_t^0 = q_t^0 = v_\alpha^0 = v_\beta^0 = 0 \quad \text{in OABCD}.
\end{aligned} \tag{35}$$

In the region ACD, we use a cylindrical polar coordinate system

$$x = a + r\ \sin(\theta) + ut, \qquad y = -r\ \cos(\theta).$$

The α^0-lines in this region are the logarithmic spirals $r = r_0 \exp[\theta\ \tan(\delta)]$ with r_0 being constant for each α^0-line, and the β^0-lines are the radial straight lines $\theta =$ constant. Accordingly, the differential operators are transformed as follows

$$\frac{\partial}{\partial s_\alpha^0} = \sin(\delta)\frac{\partial}{\partial r} + \frac{\cos(\delta)}{r}\frac{\partial}{\partial\theta}, \qquad \frac{\partial}{\partial s_\beta^0} = -\frac{\partial}{\partial r}.$$

4.2. The first-order governing equations and solutions

Now we consider the first-order problem for which the domain OA'B'C'D' is a perturbation of the initial slipline field for $\mathbf{B}_0$, OABCD. The origins of the colinear coordinate system $s_\alpha^0 s_\beta^0$ for A'B'C' and the polar coordinate system for A'C'D' coincide with point A', and the s_α^0- and s_β^0-axes are parallel to the α^0- and β^0-lines in ABC. The origin of the colinear coordinate system in OA'D' is at O and the axes are parallel to the α^0- and β^0-lines in OAD. These coordinate systems translate in the x-direction with speed u; thus, on the contact surface OA', we have

$$g'(x,t) = \frac{(x-ut)^2}{b},$$

and

$$\lambda' = \frac{2(x-ut)}{b}, \qquad \lambda_t' = -\frac{2u}{b}.$$

The first-order boundary conditions on OA' are

$$\psi' = \lambda' = \frac{2(x-ut)}{b}, \qquad \psi_t' = -\frac{2u}{b},$$

$$(v_\alpha' - v_\beta')\ \sin\left(\frac{\pi}{4}+\frac{\delta}{2}\right) = u\lambda' = \frac{2u(x-ut)}{b}.$$

Next we set up another Cartesian coordinate system O'$x'y'$ which coincides with system Oxy at the start of the rolling and then translates in the x-direction with speed u, hence

$$x' = x - ut, \qquad y' = y.$$

Relative to the moving system O'$x'y'$, the boundary conditions are

$$\psi' = \frac{2x'}{b}, \qquad \psi'_t = -\frac{2u}{b}, \qquad (v'_\alpha - v'_\beta)\,\sin\left(\frac{\pi}{4} + \frac{\delta}{2}\right) = \frac{2ux'}{b}. \tag{36}$$

The profile of the traction free surface A'B' is unknown initially and is determined as part of the solution; let it be given by $y' = \varepsilon g'(x', t)$, such that

$$\lambda' = \frac{\partial g'}{\partial x'} \equiv G(x', t), \qquad \lambda'_t = \frac{\partial G}{\partial t} \equiv F(x', t),$$

wherefrom the conditions on A'B' may be expressed as

$$\psi' = G(x', t), \qquad q' = 0, \qquad \psi'_t = F(x', t), \qquad q'_t = 0. \tag{37}$$

Note that in consequence of the assumed steady-state conditions, the perturbed slipline field illustrated in Figure 2 remains unchanged with respect to the moving coordinate system O'$x'y'$. Hence, in all the calculations presented hereafter, the perturbed slipline field may be envisaged as being the same as that shown in Figure 2 with the coordinate system Oxy changed to O'$x'y'$.

With the zeroth-order solutions given in (35), the first-order governing equations for stresses are

$$\cot(\delta)\frac{\partial q'}{\partial s^0_\alpha} + 2q^0\frac{\partial \psi'}{\partial s^0_\alpha} = 0,$$
$$\cot(\delta)\frac{\partial q'}{\partial s^0_\beta} - 2q^0\frac{\partial \psi'}{\partial s^0_\beta} = 0 \quad \text{in A'B'C' and O'A'D';}$$

and

$$\begin{aligned}
&\cot(\delta)\left[\sin(\delta)\frac{\partial q'}{\partial r} + \frac{\cos(\delta)}{r}\frac{\partial q'}{\partial \theta}\right] + 2q^0\left[\sin(\delta)\frac{\partial \psi'}{\partial r} + \frac{\cos(\delta)}{r}\frac{\partial \psi'}{\partial \theta}\right] \\
&\quad + 2q'\frac{\cos(\delta)}{r} = 0, \\
&\cot(\delta)\frac{\partial q'}{\partial r} - 2q^0\left[\frac{\partial \psi'}{\partial r} + \frac{2}{r}\psi'\right] = 0 \quad \text{in A'C'D'.}
\end{aligned} \tag{38}$$

The equations for the first-order stress rates are

$$
\begin{aligned}
&\cot(\delta)\frac{\partial q_t'}{\partial s_\alpha^0} + 2q^0\frac{\partial \psi_t'}{\partial s_\alpha^0} = 0,\\
&\cot(\delta)\frac{\partial q_t'}{\partial s_\beta^0} - 2q^0\frac{\partial \psi_t'}{\partial s_\beta^0} = 0 \quad \text{in A'B'C' and O'A'D';}
\end{aligned}
$$

and

$$
\begin{aligned}
&\cot(\delta)\left[\sin(\delta)\frac{\partial q_t'}{\partial r} + \frac{\cos(\delta)}{r}\frac{\partial q_t'}{\partial \theta}\right] + 2q^0\left[\sin(\delta)\frac{\partial \psi_t'}{\partial r} + \frac{\cos(\delta)}{r}\frac{\partial \psi_t'}{\partial \theta}\right]\\
&\quad +2q_t'\frac{\cos(\delta)}{r} = 0,\\
&\cot(\delta)\frac{\partial q_t'}{\partial r} - 2q^0\left[\frac{\partial \psi_t'}{\partial r} + \frac{2}{r}\psi_t'\right] = 0 \quad \text{in A'C'D'.}
\end{aligned}
\tag{39}
$$

The first-order velocity field satisfy

$$
\begin{aligned}
&\frac{\partial v_\alpha'}{\partial s_\alpha^0} - \psi_t'\tan(\delta) = 0, \qquad \frac{\partial v_\beta'}{\partial s_\beta^0} + \psi_t'\tan(\delta) = 0 \quad \text{in A'B'C' and O'A'D';}
\end{aligned}
$$

and

$$
\begin{aligned}
&\sin(\delta)\frac{\partial v_\alpha'}{\partial r} + \frac{\cos(\delta)}{r}\frac{\partial v_\alpha'}{\partial \theta} - v_\beta'\frac{1}{r} - \psi_t'\tan(\delta) = 0 \quad \text{in A'C'D',}\\
&\frac{\partial v_\beta'}{\partial r} - v_\alpha'\frac{\sin(\delta)}{r} - \psi_t'\tan(\delta) = 0.
\end{aligned}
\tag{40}
$$

4.2.1. *First-order stresses*

Integrating the Equations $(38)_{1,2}$ and applying the conditions $(37)_{1,2}$ we obtain the solutions for the first-order stresses at a point $P(s_\alpha^0, s_\beta^0)$ in region A'B'C'

$$
q' = q_I^0\tan(\delta)\,[G(x_S', t) - G(x_R', t)], \qquad \psi' = \tfrac{1}{2}[G(x_S', t) + G(x_R', t)], \tag{41}
$$

with

$$
x_S' = a - 2s_\beta^0\cos\left(\frac{\pi}{4} - \frac{\delta}{2}\right), \qquad x_R' = a + 2s_\alpha^0\cos\left(\frac{\pi}{4} - \frac{\delta}{2}\right).
$$

For simplicity, we approximate the profile of the traction free surface by

$$
y' = \varepsilon[l x'^2 + m x' + n], \tag{42}
$$

with l, m and n being constant. Then the first-order stresses are

$$
\begin{aligned}
&q' = -4\,l\,q_I^0\tan(\delta)(s_\beta^0 + s_\alpha^0)\cos\left(\frac{\pi}{4} - \frac{\delta}{2}\right),\\
&\psi' = -2l\,(s_\beta^0 - s_\alpha^0)\cos\left(\frac{\pi}{4} - \frac{\delta}{2}\right) + 2la + m.
\end{aligned}
\tag{43}
$$

Since the stresses must be continuous along the interface between the regions A'B'C' and A'C'D', and the first-order stresses from the A'B'C' side are linear functions of r, we seek solutions in the region A'C'D' of the form

$$q' = L(\theta)r + M(\theta), \qquad \psi' = K(\theta)r + N(\theta).$$

It follows from $(38)_4$ that

$$N(\theta) = 0, \qquad L(\theta) = 6\ \tan(\delta) q^0_{II}(\theta)\ K(\theta).$$

Equation $(38)_3$ then yields

$$M(\theta) = A'\ \exp[-2\theta\ \tan(\delta)], \qquad K(\theta) = B'\ \exp[-\theta\ \tan(\delta)],$$

where A' and B' are arbitrary constants. Thus the general solutions for the first-order stress field in region A'C'D' are

$$q' = 6B'\ \tan(\delta)\, q^0_{II}(\theta)\ \exp[-\theta\ \tan(\delta)]r + A' \exp[-2\theta\ \tan(\delta)],$$
$$\psi' = B'\ \exp[-\theta\ \tan(\delta)]r.$$

To satisfy the continuity conditions of stresses q and ψ along A'C', we first need to find the equation for this interfacial β-line. To order ε, the differential equation for A'C' is

$$r\,\frac{\mathrm{d}\theta}{\mathrm{d}r} = \varepsilon\psi' = \varepsilon B'\ \exp[-\theta\ \tan(\delta)]r;$$

this line is inclined to the traction free surface at angle $\pi/4 - \delta/2$ at point A', so $\theta \to \pi/4 + \delta/2 + \varepsilon G(a^+)$ when $r \to 0$ with $G(a^+) = 2la + m$. Hence the solution to the above differential equation is given by

$$\exp[\theta\ \tan(\delta)] = \varepsilon B' r\ \tan(\delta) + \exp\left[\left(\frac{\pi}{4} + \frac{\delta}{2} + \varepsilon G(a^+)\right)\tan(\delta)\right],$$

or, to order ε,

$$\theta = \frac{\pi}{4} + \frac{\delta}{2} + \varepsilon G(a^+) + \varepsilon B' r\ \exp\left[-\left(\frac{\pi}{4} + \frac{\delta}{2}\right)\tan(\delta)\right].$$

Along this interface, we have from the side of region A'B'C': $s^0_\beta = -r$, $s^0_\alpha = 0$ and

$$q = q^0 + \varepsilon q' = q^0_I + 4\varepsilon\ \tan(\delta) q^0_I\ \cos\left(\frac{\pi}{4} - \frac{\delta}{2}\right) l\, r,$$
$$\psi = \psi^0 + \varepsilon\psi' = \frac{\pi}{2} + \varepsilon\left[2l\ \cos\left(\frac{\pi}{4} - \frac{\delta}{2}\right) r + 2l\, a + m\right];$$

whereas from the side of region A'C'D', we have

$$\begin{aligned} q = q^0 + \varepsilon q' \ &=\ q^0_I + 4\varepsilon B'\ \tan(\delta) q^0_I\ \exp\left[-\left(\frac{\pi}{4} + \frac{\delta}{2}\right)\tan(\delta)\right] r \\ &\quad + \varepsilon A'\ \exp\left[-\left(\frac{\pi}{2} + \delta\right)\tan(\delta)\right] - 2\varepsilon q^0_I\ \tan(\delta) G(a^+), \\ \psi = \psi^0 + \varepsilon\psi' \ &=\ \frac{\pi}{2} + 2\varepsilon B'\ \exp\left[-\left(\frac{\pi}{4} + \frac{\delta}{2}\right)\tan(\delta)\right] r + \varepsilon[2l\, a + m]. \end{aligned}$$

Consequently, the continuity conditions lead to

$$\begin{aligned} A' &= 2[2l\,a + m]q_I^0 \tan(\delta) \exp\left[2\left(\frac{\pi}{4} + \frac{\delta}{2}\right)\tan(\delta)\right], \\ B' &= l\,\cos\left(\frac{\pi}{4} - \frac{\delta}{2}\right) \exp\left[\left(\frac{\pi}{4} + \frac{\delta}{2}\right)\tan(\delta)\right], \end{aligned} \tag{44}$$

and the first order stresses in region A'C'D' are

$$\begin{aligned} q' &= 6l\,q_I^0 \tan(\delta) \cos\left(\frac{\pi}{4} - \frac{\delta}{2}\right) \exp\left[3\left(\frac{\pi}{4} + \frac{\delta}{2} - \theta\right)\tan(\delta)\right] r \\ &\quad + 2[2l\,a + m]q_I^0 \tan(\delta) \exp\left[2\left(\frac{\pi}{4} + \frac{\delta}{2} - \theta\right)\tan(\delta)\right], \\ \psi' &= l\,\cos\left(\frac{\pi}{4} - \frac{\delta}{2}\right) \exp\left[\left(\frac{\pi}{4} + \frac{\delta}{2} - \theta\right)\tan(\delta)\right] r. \end{aligned} \tag{45}$$

The constants l, m and a are to be determined when the velocity is found in the whole plastic region.

In the region O'A'D', the general solutions to the Equations $(38)_{1,2}$ for the first-order stresses are

$$q' = \tfrac{1}{2}\tan(\delta)[I(s_\beta^0) + J(s_\alpha^0)], \qquad \psi' = \frac{1}{4q_{III}^0}[I(s_\beta^0) - J(s_\alpha^0)],$$

where $I(s_\beta^0)$ and $J(s_\alpha^0)$ are two arbitrary functions. The condition $(36)_1$ on the contact surface where $s_\beta^0 = s_\alpha^0 + O(\varepsilon)$ yields

$$J(s_\alpha^0) = I(s_\alpha^0) - 16q_{III}^0 \frac{\cos\left(\dfrac{\pi}{4} + \dfrac{\delta}{2}\right) s_\alpha^0}{b}.$$

The continuity conditions on the interfacial β-line A'D' will determine the function $I(s_\alpha^0)$. The equation for the β-line A'D' (inclined to the contact surface at angle $\pi/4 + \delta/2$) is, to order ε,

$$\theta = -\left(\frac{\pi}{4} - \frac{\delta}{2}\right) + \varepsilon\left\{2\frac{a}{b} + B'r\,\exp\left[\left(\frac{\pi}{4} - \frac{\delta}{2}\right)\tan(\delta)\right]\right\}.$$

After applying the continuity conditions, we find the first-order stresses in the region O'A'D' to be

$$\begin{aligned} q' &= 4B'q_{III}^0 \tan(\delta) \exp\left[\left(\frac{\pi}{4} - \frac{\delta}{2}\right)\tan(\delta)\right](2r_E - s_\alpha^0 - s_\beta^0) \\ &\quad - 8q_{III}^0 \tan(\delta)\frac{1}{b}\cos\left(\frac{\pi}{4} + \frac{\delta}{2}\right)s_\alpha^0 + A' \exp\left[2\left(\frac{\pi}{4} - \frac{\delta}{2}\right)\tan(\delta)\right], \\ \psi' &= 2B' \exp\left[\left(\frac{\pi}{4} - \frac{\delta}{2}\right)\tan(\delta)\right](s_\alpha^0 - s_\beta^0) + \frac{4}{b}\cos\left(\frac{\pi}{4} + \frac{\delta}{2}\right)s_\alpha^0, \end{aligned} \tag{46}$$

where A' and B' are given in (44) and $r_E = a/[2\cos(\pi/4 + \delta/2)]$.

4.2.2. *First-order stress rates*

Now we consider the first-order stress rate field. Based on Equation (42), we have $F(x', t) = 0$, so $\psi_t' = 0$ on the free surface A'B'. The solutions to the Equations $(39)_{1,2}$ in region A'B'C' satisfying the conditions $(36)_{3,4}$ are

$$\psi_t' = q_t' = 0. \tag{47}$$

Since $\psi_t^0 = q_t^0 = 0$ in the whole plastic region, the continuity of the stress rate combination $\cot(\delta)q_t + 2q\psi_t$ along the interfacial β-lines A'C' and A'D' reduces to the continuity of $\cot(\delta)q_t' + 2q^0\psi_t'$. Along A'C', this combination is zero from the side of the region A'B'C'. To satisfy the continuity condition, we seek a solution for the first-order stress rates inside the region A'C'D' of form

$$q_t' = -2\tan(\delta)q_{II}^0\psi_t'.$$

Substituting this in the Equations $(39)_{3,4}$, we obtain

$$\psi_t' = \frac{P(\theta)}{r}, \qquad q_t' = -\frac{2\tan(\delta)q_{II}^0 P(\theta)}{r}, \tag{48}$$

where $P(\theta)$ is an arbitrary function.

The general solutions to the Equations $(39)_{1,2}$ for the stress rates in region O'A'D' are

$$\cot(\delta)q_t' + 2q_{III}^0\psi_t' = Q(s_\beta^0), \qquad \cot(\delta)q_t' - 2q_{III}^0\psi_t' = S(s_\alpha^0),$$

with two arbitray functions $Q(s_\beta^0)$ and $S(s_\alpha^0)$. The continuity of $\cot(\delta)q_t' + 2q^0\psi_t'$ along A'D' leads to $Q(s_\beta^0) \equiv 0$, and the condition $(36)_2$ on the contact surface yields $S(s_\alpha^0) = 8q_{III}^0 u/b$. Thus we have the first-order stress rates in the region O'A'D' as follows

$$\psi_t' = -\frac{2u}{b}, \qquad q_t' = \frac{4\tan(\delta)\, q_{III}^0 u}{b}. \tag{49}$$

4.2.3. *First-order velocity*

For the velocity field we work through O'A'D', A'C'D' to A'B'C'. Since $v_\alpha^0 = v_\beta^0 = 0$ in the whole plastic region, the continuity conditions for the velocity reduces to the continuity of the first-order velocity. Therefore, along the bounding α-line O'D'C'B', we must have $v_\beta' = 0$ since the material outside the plastic region is stationary.

With ψ_t' given in $(49)_1$, the general solutions to the Equations $(40)_{1,2}$ for the first-order velocity field are

$$v_\alpha' = -2\frac{u}{b}\tan(\delta)s_\alpha^0 + I_v(s_\beta^0), \qquad v_\beta' = 2\frac{u}{b}\tan(\delta)s_\beta^0 + J_v(s_\alpha^0).$$

The condition $v_\beta' = 0$ on O'D' (where $s_\beta^0 = 0$) leads to $J_v(s_\alpha^0) \equiv 0$. Then, applying the condition $(36)_3$ on the contact surface, we obtain $I_v(s_\beta^0) = 4u\sec(\delta)s_\beta^0/b$, and therefore the first-order velocity field in region O'A'D' is

$$v_\alpha' = -2\frac{u}{b}[\tan(\delta)s_\alpha^0 - 2\sec(\delta)s_\beta^0], \qquad v_\beta' = 2\frac{u}{b}\tan(\delta)s_\beta^0. \tag{50}$$

Substituting $\psi_t' = P(\theta)/r$ in the Equations $(40)_{3,4}$, we have the following governing equations for the first-order velocity in the region A'C'D'

$$\begin{aligned} &\sin(\delta)\frac{\partial v_\alpha'}{\partial r} + \frac{\cos(\delta)}{r}\frac{\partial v_\alpha'}{\partial \theta} - v_\beta'\frac{1}{r} - \frac{1}{r}P(\theta)\ \tan(\delta) = 0, \\ &\frac{\partial v_\beta'}{\partial r} - v_\alpha'\frac{\sin(\delta)}{r} - \frac{1}{r}P(\theta)\ \tan(\delta) = 0. \end{aligned} \tag{51}$$

For $v_\beta' = 0$ on D'C' where $r = r_E\ \exp[(\pi/4 - \delta/2 + \theta)\ \tan(\delta)]$, we seek a partial solution of the form

$$v_\beta' = H(\theta)\left\{r - r_E\ \exp\left[\left(\frac{\pi}{4} - \frac{\delta}{2} + \theta\right)\ \tan(\delta)\right]\right\},$$

with $H(\theta)$ being arbitrary. Then Equation $(50)_2$ requires

$$v_\alpha' = \mathrm{cosec}(\delta)H(\theta)r - \sec(\delta)P(\theta).$$

The general solutions for $H(\theta)$ and $P(\theta)$ which satisfy Equation $(50)_1$ are

$$H(\theta) = H_0,$$

$$P(\theta) = P_0\ \exp[-\theta\ \tan(\delta)] + \tfrac{1}{2}H_0 r_E\ \cot(\delta)\ \exp\left[\left(\frac{\pi}{4} - \frac{\delta}{2} + \theta\right)\ \tan(\delta)\right],$$

where H_0 and P_0 are arbitrary constants. The continuity of v_α' along A'D' leads to

$$H_0 = -\frac{4u\tan(\delta)}{b}, \qquad P_0 = -\frac{2ur_E[1 - \sin(\delta)]\ \exp\left[-\left(\frac{\pi}{4} - \frac{\delta}{2}\right)\ \tan(\delta)\right]}{b}.$$

Hence we have the velocity components in region A'C'D'

$$\begin{aligned} v_\alpha' \ &=\ -4\frac{u}{b}\sec(\delta)\,r + 2\frac{u}{b}r_E\ \sec(\delta)\left\{[1 - \sin(\delta)]\ \exp\left[-\left(\frac{\pi}{4} - \frac{\delta}{2} + \theta\right)\ \tan(\delta)\right]\right. \\ &\qquad \left. + \exp\left[\left(\frac{\pi}{4} - \frac{\delta}{2} + \theta\right)\ \tan(\delta)\right]\right\}, \\ v_\beta' \ &=\ -4\frac{u}{b}\tan(\delta)\left\{r - r_E\ \exp\left[\left(\frac{\pi}{4} - \frac{\delta}{2} + \theta\right)\ \tan(\delta)\right]\right\}. \end{aligned} \tag{52}$$

In the region A'B'C', the first-order stress rates vanish. So the general solutions to Equations $(40)_{1,2}$ are

$$v_\alpha' = F^*(s_\beta^0), \qquad v_\beta' = G^*(s_\alpha^0),$$

with $F^*(s_\beta^0)$ and $G^*(s_\alpha^0)$ arbitrary. Applying the condition $v_\beta' = 0$ on C'B' and the condition of continuity of v_α' across the β-line A'C', we obtain the first-order velocity components in this region

$$v_\alpha' = 4\frac{u}{b}\sec(\delta)s_\beta^0 + 2\frac{u}{b}r_E\ \sec(\delta)\bar{V}, \qquad v_\beta' = 0, \tag{53}$$

with

$$\bar{V} = \exp\left[\frac{\pi}{2}\tan(\delta)\right] + [1-\sin(\delta)]\,\exp\left[-\frac{\pi}{2}\tan(\delta)\right].$$

5. Results and discussion

Relative to the coordinate system O'$x'y'$, the free-surface profile remains unchanged, hence the relative movement of particles is directed along this surface. Accordingly, the slope of the free-surface profile may be expressed as follows

$$\frac{\mathrm{d}y'}{\mathrm{d}x'} = \frac{v_{y'}}{v_{x'}-u} = -\varepsilon\frac{v'_{y'}}{u-\varepsilon v'_{x'}} = -\varepsilon\frac{v'_{y'}}{u} + O(\varepsilon^2), \tag{54}$$

where

$$\begin{aligned} v'_{y'} &= v'_\alpha \sin\left(\frac{\pi}{4}-\frac{\delta}{2}\right) \\ &= 2\frac{u}{b}\sec(\delta)\left[\tan\left(\frac{\pi}{4}-\frac{\delta}{2}\right)(a-x') + \sin\left(\frac{\pi}{4}-\frac{\delta}{2}\right) r_E\bar{V}\right]. \end{aligned}$$

Substituting this in (54), integrating it and making use of the condition that $y' = \varepsilon a^2/b$ when $x' = a$, we obtain the expression for the profile of the traction free surface, to order ε, as follows

$$y' = \varepsilon\frac{1}{b}\sec(\delta)\,\tan\left(\frac{\pi}{4}-\frac{\delta}{2}\right)\left\{x'^2 - 2\left[a + r_E\bar{V}\,\cos\left(\frac{\pi}{4}-\frac{\delta}{2}\right)\right]x' + N_0\right\}, \tag{55}$$

where

$$N_0 = a^2\left\{1+\cot\left(\frac{\pi}{4}-\frac{\delta}{2}\right)[\bar{V}+\cos(\delta)]\right\}.$$

By comparing the two expressions in (42) and (55) for the free surface, we get

$$l = \frac{1}{b}\sec(\delta)\,\tan\left(\frac{\pi}{4}-\frac{\delta}{2}\right), \qquad m = -\frac{1}{b}\sec(\delta)\left[2\tan\left(\frac{\pi}{4}-\frac{\delta}{2}\right)+\bar{V}\right]a. \tag{56}$$

Substituting these in (44), and combining the results with $(46)_1$ and $(30)_1$, we obtain the following expression for the normal pressure $-\sigma_n$ on the contact surface

$$\begin{aligned} -\sigma_n &= -(\sigma_n^0 + \varepsilon\sigma'_n) \\ &= c\,\cot(\delta)[Q_0-1] + \frac{2}{b}\varepsilon c Q_0\{2Q_1(a-x') - 2x' - \sec(\delta)\bar{V}a\}, \end{aligned} \tag{57}$$

with

$$Q_0 = \frac{1+\sin(\delta)}{1-\sin(\delta)}\exp[\pi\,\tan(\delta)], \qquad Q_1 = \sec(\delta)\,\exp\left[\frac{\pi}{2}\tan(\delta)\right].$$

Denoting the vertical load by W and the rolling resistance by F (the minimum value required for the translatory force to cause forward movement of the cylinder), we have

$$
\begin{aligned}
W &= -\int_0^{s_a} \sigma_n \,\mathrm{d}s\ \cos(\lambda) = -\int_0^{a} \sigma_n \,\mathrm{d}x', \\
F &= -\int_0^{s_a} \sigma_n \,\mathrm{d}s\ \sin(\lambda) = -\int_0^{a} \sigma_n\ \tan(\lambda)\mathrm{d}x',
\end{aligned}
\tag{58}
$$

where s_a is the arc length of the contact surface and λ is the angle between the contact surface and the x' axis. Substituting (57) in $(58)_1$ and integrating, we get

$$
W = c\ \cot(\delta)[Q_0 - 1]a + \frac{2\varepsilon\, c\ Q_0\{Q_1 - 1 - \sec(\delta)\bar{V}\}a^2}{b}.
$$

According to the definition of $\varepsilon = a/R$ and $b = 2R\varepsilon = 2a$, this equation can be expressed as

$$
W = c\ \cot(\delta)[Q_0 - 1]a + \frac{c\ Q_0\{Q_1 - 1 - \sec(\delta)\bar{V}\}a^2}{R}. \tag{59}
$$

For a given vertical load W, the contact width a may then be determined from this equation. With the approximation of $y' = x'^2/2R$, $\tan(\lambda) = x'/R$, the rolling resistance F is obtained by integrating $(58)_2$ as follows

$$
F = \tfrac{1}{2}c\ \cot(\delta)[Q_0 - 1]\frac{a^2}{R} + \tfrac{1}{3}c\ Q_0\{Q_1 - 2 - \tfrac{3}{2}\sec(\delta)\bar{V}\}\frac{a^3}{R^2}. \tag{60}
$$

Once the contact width a is established, (60) can be used to calculate the rolling resistance. In addition, with $b = 2a$, the profile of traction free surface is given by (55).

The passage of the cylinder leads to a permanent horizontal displacement Δ on the surface of the granular material in the direction of motion of the cylinder, namely

$$
\Delta = \frac{\int_0^{\eta a} v_{x'}\,\mathrm{d}x'}{u} = \frac{\varepsilon \int_0^{\eta a} v'_{x'}\,\mathrm{d}x'}{u}, \tag{61}
$$

where

$$
\eta a = \left\{1 + \tan\left(\frac{\pi}{4} + \frac{\delta}{2}\right)\ \exp\left[\frac{\pi}{2}\tan(\delta)\right]\right\} a \tag{62}
$$

is the length of O’B. Substituting the corresponding expressions for $v'_{x'}$ for the contact and free surfaces in the above, we find that

$$
\Delta = \frac{a^2}{2R}\sec(\delta)\left\{1 + [1 - \sin(\delta)]\ \tan^2\left(\frac{\pi}{4} + \frac{\delta}{2}\right)\right\}. \tag{63}
$$

Because of the steady-state and incompressibility conditions assumed for the fully developed flow regime, the surface levels of the material both ahead and behind the rolling cylinder are the same. Therefore, the permanent vertical displacements which are considered to occur at the initial stages of loading are outside the scope of this model (recall that the difference between

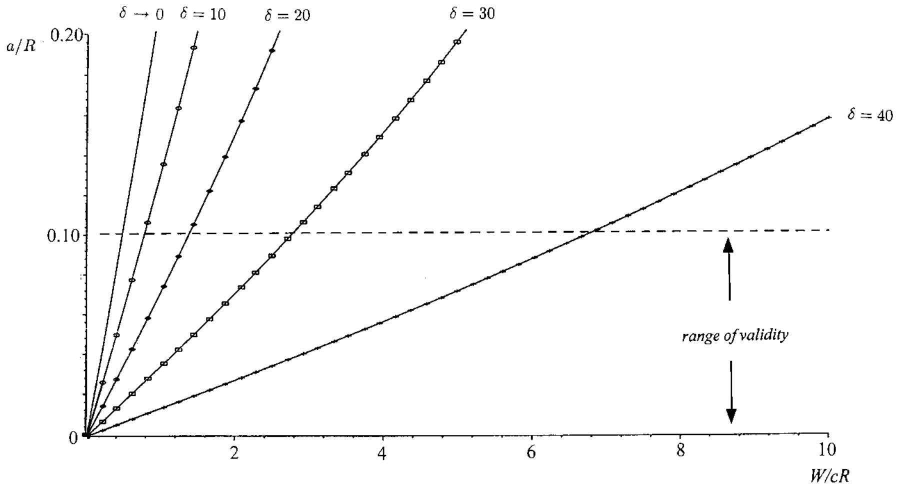

Figure 3. Contact width a/R as a function of vertical load W/cR for various values of internal friction angle δ. Note that the horizontal dashed line indicates the upper limit of the range of validity: $\varepsilon(= a/R) < 0{\cdot}1$ (*e.g.*, for $\delta = 30°$, range of validity lies in $0 \leqslant W/cR < 2{\cdot}7871$)

the initial stages of loading and the fully developed flow regime is illustrated in Figure 1). Moreover, in consequence of the steady-state condition, the contact width, the volume of bulldozed material, and the free-surface profile are independent of the speed of the cylinder.

In the following, we summarise some trends which arise from changes in material property via the parameter δ. Figures 3, 4 and 5 present, respectively, the variations of the contact width a/R, permanent horizontal displacement Δ/R, and coefficient of rolling resistance F/W, with the vertical load W for several values of internal friction angle of the material δ. Bearing in mind that the perturbation approximation is generally applicable when $\varepsilon = a/R \ll 1{\cdot}0$ (*e.g.*, $\varepsilon < 0{\cdot}1$), Figure 3 indicates that both the range of validity and the accuracy of the perturbation solutions increase with increasing δ. For a fixed load W, Figure 3 shows a decrease in the contact width with increasing δ. Under these conditions, one finds a concomitant decrease in the overall plastic region based on consideration of the location of the perturbed bounding α-line OD'C'B' which may be defined as follows

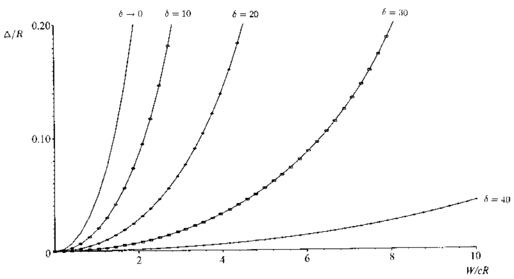

Figure 4. Permanent surface displacement Δ/R as a function of vertical load W/cR for various values of internal friction angle δ.

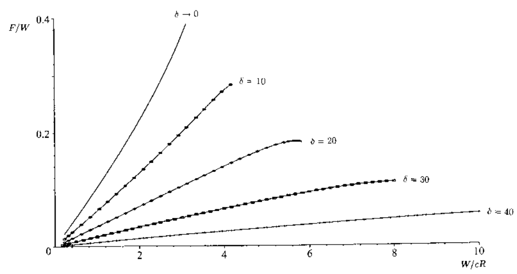

Figure 5. Coefficient of rolling resistance F/W as a function of vertical load W/cR for various values of internal friction angle δ.

Table 1. Perturbed and zeroth-order solutions: the maximum horizontal extent of the slipline field (columns 3–4) and the contact pressures (columns 5–7) for varying values of internal friction angle δ for $W/cR = 0{\cdot}4$.

δ	a/R	(length $O'B'$)$/R$	(length $O'B$)$/R$	$-\sigma_n^0/c$	$-(\sigma_n^0 + \varepsilon\sigma_n')/c$	$-(\varepsilon\sigma_n')/\sigma_n^0$
0	0·08026873	0·1544146851	0·1605954924	5·14390617	4·98326062	0·03123026387
10	0·04910154	0·1210818308	0·1419629656	8·34492612	8·14638326	0·02379204586
20	0·02741621	0·09287495284	0·1179176264	14·8347117	14·5899098	0·01650196807
30	0·01340689	0·06830379235	0·0907148111	30·1396278	29·8354152	0·01009344349
40	0·00533848	0·04658427028	0·0627183123	75·3131142	74·9276362	0·00511833900
50	0·00150164	0·02761509285	0·0365851105	266·881763	266·375832	0·00189571214

$$
\begin{aligned}
y' &= -\tan\left(\frac{\pi}{4}+\frac{\delta}{2}\right)x' + \frac{2+\sec(\delta)\,\exp\left[\frac{\pi}{2}\tan(\delta)\right]}{2R\,\cos^2\left(\frac{\pi}{4}+\frac{\delta}{2}\right)}x'^2 \quad \text{for OD'},\\
r &= r_{D'}\,\exp\left\{(\theta-\theta_{D'})\,\tan\left[\delta - \frac{a}{4R}\sec(\delta)\,\exp\left(\frac{\pi}{2}\tan(\delta)\right)\right]\right\} \quad \text{for D'C'},\\
y' &= (x'-x'_{C'})\,\tan\left(\frac{\pi}{4}-\frac{\delta}{2}\right) + \frac{\sec(\delta)}{2R}\frac{\tan\left(\frac{\pi}{4}-\frac{\delta}{2}\right)}{\cos^2\left(\frac{\pi}{4}-\frac{\delta}{2}\right)}(x'^2 - x'^2_{C'})\\
&\qquad - \frac{a\,\sec(\delta)}{2R}\frac{2\tan\left(\frac{\pi}{4}-\frac{\delta}{2}\right)+\bar{V}}{\cos^2\left(\frac{\pi}{4}-\frac{\delta}{2}\right)}(x'-x'_{C'}) + y'_{C'} \quad \text{for C'B'},
\end{aligned}
\tag{64}
$$

where the polar coordinate system (r, θ) with

$$
x' = a + r\,\sin(\theta), \qquad y' = \frac{a^2}{2R} - r\,\cos(\theta)
$$

is used in the description of D'C', while $(r_{D'}, \theta_{D'})$ and $(x'_{C'}, y'_{C'})$ represent the location of the points D' and C', respectively (see Figure 2). The decrease in the overall extent of the slipline region with increasing δ is evident in the values of the length of O'B' (*i.e.*, the maximum horizontal extent of the perturbed slipline field) as listed in column 3 of Table 1 for $W/cR = 0{\cdot}4$.

Comparisons between the zeroth-order and the perturbed solutions may be made for the overall dimensions of the slipline field and the contact pressures. It is evident from the values of O'B in column 4 of Table 1 that the curvature of the cylinder effectively reduces the slipline field; indeed an examination of the bounding α-lines O'D'C'B' (Equation (64)) and the interfacial β-lines A'D' and A'C' (in Section 4.2.1) indicates that the greater the cylinder's radius of curvature (*i.e.* the smaller the cylinder radius R), the smaller the overall slipline field will be. The zero-order contact pressure (constant), and the mean of the contact pressure distribution to first order (linear with respect to x in Equation (57)) are given in columns 5 and 6 of Table 1. It can be seen from column 7 that for a fixed load, the relative difference between the zero-order contact pressure and the first-order mean contact pressure decreases with increasing δ. This trend is consistent with Figure 3 in which, for a fixed load, the perturbation parameter $\varepsilon(= a/R)$ decreases with increasing δ.

Finally, we note that the formulation in this paper is not suitable for purely frictional materials, *i.e.* $c = 0$; this is conveyed in Equations (57)–(60) which show that the stresses in the material vanish for the case $c = 0$. On the other hand, in the limit $\delta \to 0$, which corresponds to purely cohesive materials, the expressions for the contact pressure, the forces F and W, and the permanent horizontal displacement tend to those values given in Marshall [6] for Tresca or von-Mises materials. Although a comparison with experimental data is required to assess the validity of this model properly, the above results do show sensible trends which lend merit to the future extension of this work to the case of frictional contact.

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